

JOURNAL OF MATHEMATICAL SCIENCES

Volume 234, Number 1

October 07, 2018

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Engl./Ukr.

This issue is a translation of *Ukrains'kiĭ Matematychnyi Visnyk* (*Ukrainian Mathematical Bulletin*) Vol. 14, No. 4, 2017. International Editorial Board of *Ukrains'kiĭ Matematychnyi Visnyk*: M. Atiyah, B. Bojarski, A. Friedman, O. Martio, L. Nirenberg, A. Samoilenko (Editor-in-Chief).

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Automorphisms of semigroups of k -linked upfamilies

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Presented by V. O. Derkach

Abstract. A family \mathcal{A} of non-empty subsets of a set X is called an *upfamily*, if, for each set $A \in \mathcal{A}$, any set $B \supset A$ belongs to \mathcal{A} . An upfamily \mathcal{L} is called *k -linked*, if $\bigcap \mathcal{F} \neq \emptyset$ for any subfamily $\mathcal{F} \subset \mathcal{L}$ of cardinality $|\mathcal{F}| \leq k$. The extension $N_k(X)$ consists of all k -linked upfamilies on X . Any associative binary operation $* : X \times X \rightarrow X$ can be extended to an associative binary operation $* : N_k(X) \times N_k(X) \rightarrow N_k(X)$. Here, we study automorphisms of the extensions of groups and finite monogenic semigroups. We also describe the automorphism groups of extensions of null semigroups, almost null semigroups, right zero semigroups and left zero semigroups.

Keywords. Semigroup, k -linked upfamily, automorphism group.

Introduction

We will investigate the automorphism groups of the extensions $N_k(S)$ of a semigroup S . The thorough study of various extensions of semigroups was started in [13] and continued in [1–10, 14–19]. The largest among these extensions is the semigroup $v(S)$ of all upfamilies on S . A family \mathcal{A} of non-empty subsets of a set X is called an *upfamily*, if, for each set $A \in \mathcal{A}$, any subset $B \supset A$ of X belongs to \mathcal{A} . Each family \mathcal{B} of non-empty subsets of X generates the upfamily $\langle \mathcal{B} \rangle := \{A \subset X : \exists B \in \mathcal{B} (B \subset A)\}$. An upfamily \mathcal{F} that is closed under taking finite intersections is called a *filter*. A filter \mathcal{U} is called an *ultrafilter*, if $\mathcal{U} = \mathcal{F}$ for any filter \mathcal{F} containing \mathcal{U} . The family $\beta(X)$ of all ultrafilters on a set X is called *the Stone–Čech compactification of X* , see [20, 24]. An ultrafilter $\langle \{x\} \rangle$ generated by a singleton $\{x\}$, $x \in X$ is called *principal*. Each point $x \in X$ is identified with the principal ultrafilter $\langle \{x\} \rangle$ generated by the singleton $\{x\}$. Hence, we can consider $X \subset \beta(X) \subset v(X)$. It was shown in [13] that any associative binary operation $* : S \times S \rightarrow S$ can be extended to an associative binary operation $* : v(S) \times v(S) \rightarrow v(S)$ by the formula

$$A * B = \left\langle \bigcup_{a \in A} a * B_a : A \in \mathcal{A}, \{B_a\}_{a \in A} \subset \mathcal{B} \right\rangle$$

for upfamilies $\mathcal{A}, \mathcal{B} \in v(S)$. In this case, the Stone–Čech compactification $\beta(S)$ is a subsemigroup of the semigroup $v(S)$.

The semigroup $v(S)$ contains many other important extensions of S . In particular, it contains the semigroups $N_k(S)$ of k -linked upfamilies for $k \in \mathbb{N} \setminus \{1\}$. An upfamily $\mathcal{L} \in v(S)$ is called *k -linked*, if $\bigcap \mathcal{F} \neq \emptyset$ for any subfamily $\mathcal{F} \subset \mathcal{L}$ of cardinality $|\mathcal{F}| \leq k$. The space $N_k(S)$ is well-known in general and categorial Topologies, see [22–25].

For a finite set X , the cardinality of the set $N_k(X)$ grows very quickly as $|X|$ tends to infinity. The calculation of the cardinality of $N_k(X)$ seems to be a difficult combinatorial problem related to the yet

unsolved Dedekind problem of calculation of the number $M(n)$ of all monotone Boolean functions of n Boolean variable, see [11].

We were able to calculate the cardinalities of $N_k(X)$ only for sets X of cardinality $|X| \leq 5$, see [12]. The results of (computer) calculations are presented in Table 1.

$ X $	$ N_2(X) $	$ N_3(X) $	$ N_4(X) $
1	1	1	1
2	3	3	3
3	11	10	10
4	80	54	53
5	2645	762	687

Table 1: The cardinalities of $N_k(X)$ for sets X of cardinality $|X| \leq 5$

Each map $f : X \rightarrow Y$ for each $k \in \mathbb{N} \setminus \{1\}$ induces the map

$$N_k f : N_k(X) \rightarrow N_k(Y), \quad N_k f : \mathcal{M} \mapsto \langle f(M) : M \in \mathcal{M} \rangle, \quad \text{see [12].}$$

If $\varphi : S \rightarrow S'$ is a homomorphism of semigroups, then, for each $k \in \mathbb{N} \setminus \{1\}$, the map $N_k \varphi : N_k(S) \rightarrow N_k(S')$ is a homomorphism as well, see [13].

Recall that an *isomorphism* between semigroups S and S' is a bijective function $\psi : S \rightarrow S'$ such that $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in S$. If there exists an isomorphism between S and S' , then S and S' are said to be *isomorphic*, which is denoted by $S \cong S'$. An isomorphism $\psi : S \rightarrow S$ is called an *automorphism* of a semigroup S . By $\text{Aut}(S)$, we denote the automorphism group of a semigroup S .

A non-empty subset I of a semigroup S is called an *ideal*, if $IS \cup SI \subset I$. An ideal I of a semigroup S is said to be *proper*, if $I \neq S$. A proper ideal M of S is *maximal*, if M coincides with each proper ideal I of S that contains M . An element z of a semigroup S is called a *zero* (resp., a *left zero*, a *right zero*) in S , if $az = za = z$ (resp., $za = z$, $az = z$) for any $a \in S$. An element e of a semigroup S is called an *idempotent*, if $ee = e$. By $E(S)$, we denote the set of all idempotents of a semigroup S .

1. Extending automorphisms from a semigroup to its extensions

In this section, we show that, for each $k \in \mathbb{N} \setminus \{1\}$, any automorphism of a semigroup S can be extended to an automorphism of its extension $N_k(S)$, and the automorphism group $\text{Aut}(N_k(S))$ of the extension $N_k(S)$ of a semigroup S contains a subgroup isomorphic to the group $\text{Aut}(S)$.

The following propositions are corollaries of the functoriality of N_k in the category of semigroups, see [3, 24].

Proposition 1.1. *If $\psi : S \rightarrow S$ is an automorphism of a semigroup S , then, for each $k \in \mathbb{N} \setminus \{1\}$, the map $N_k \psi : N_k(S) \rightarrow N_k(S)$ is an automorphism of the extension $N_k(S)$.*

Proposition 1.2. *For each $k \in \mathbb{N} \setminus \{1\}$, the automorphism group $\text{Aut}(N_k(S))$ of the extension $N_k(S)$ of a semigroup S contains a subgroup isomorphic to the automorphism group $\text{Aut}(S)$ of S .*

2. The automorphism groups of the extensions $N_k(G)$ of a group G

In this section, we study the automorphisms of extensions $N_k(G)$ of a group G .

Proposition 2.1. *Let G be a group, $k \in \mathbb{N} \setminus \{1\}$. If $\psi : N_k(G) \rightarrow N_k(G)$ is an automorphism, then $\psi(G) = G$.*

Proof. It was shown in [8, Proposition 4.2] that $N_k(G) \setminus G$ is an ideal of $N_k(G)$. Let us prove that $N_k(G) \setminus G$ is the unique maximal ideal of $N_k(G)$. Indeed, let I be any ideal of $N_k(G)$. If $g \in G \cap I$, then $N_k(G) = gN_k(G) \subset I$ and, hence, $I = N_k(G)$. Consequently, $N_k(G) \setminus G$ contains each proper ideal of $N_k(G)$. Taking into account that the set of maximal ideals of a semigroup is preserved by isomorphisms and $N_k(G) \setminus G$ is the unique maximal ideal of $N_k(G)$, we conclude that $\psi(N_k(G) \setminus G) = N_k(G) \setminus G$. Therefore, $\psi(G) = G$. \square

Corollary 2.2. *Each automorphism of $N_k(G)$ is an extension of an automorphism of a group G .*

Next, we will describe the structure of the automorphism groups of extensions $N_k(G)$ of finite groups G of cardinality $|G| \leq 3$.

Before describing the structure of extensions of finite groups, let us make some remarks concerning the structure of a semigroup S containing a group G with the identity element, which is also a left identity of S . In this case, S can be considered as a G -space endowed with the left action of the group G . So, we can consider the orbit space $S/G = \{Gs : s \in S\}$ and the projection $\pi : S \rightarrow S/G$. If G lies in the center of the semigroup S (which means that the elements of G commute with all the elements of S), then the orbit space S/G admits a unique semigroup operation turning S/G into a semigroup and the orbit projection $\pi : S \rightarrow S/G$ into a semigroup homomorphism. If $s \in S$ is an idempotent, then the orbit Gs is a group isomorphic to a quotient group of G . A subsemigroup $T \subset S$ will be called a *transversal semigroup*, if the restriction $\pi : T \rightarrow S/G$ is an isomorphism of the semigroups. If S admits a transversal semigroup T , then it is a homomorphic image of the product $G \times T$ under the semigroup homomorphism

$$h : G \times T \rightarrow S, \quad h : (g, t) \mapsto gt.$$

This helps one to recover the algebraic structure of S from the structure of a transversal semigroup.

2.1. The semigroups $N_k(C_1)$

For the cyclic group C_1 , the semigroups $N_k(C_1)$, $k \geq 2$, are isomorphic to C_1 . Therefore, $\text{Aut}(N_k(C_1)) \cong \text{Aut}(C_1) \cong C_1$.

2.2. The semigroups $N_k(C_2)$

For the cyclic group C_2 , the semigroups $N_k(C_2)$, $k \geq 2$, contain two principal ultrafilters and the k -linked upfamily $\{C_2\}$, which is the zero in $N_k(C_2)$. The semigroups $N_k(C_2)$ are isomorphic to the semigroup $\{-1, 0, 1\}$. Since the zero is preserved by the automorphisms of semigroups, each automorphism of C_2 is extended to the unique automorphism of $N_k(C_2)$ by Corollary 2.2. Therefore, $\text{Aut}(N_k(C_2)) \cong \text{Aut}(C_2) \cong C_1$.

2.3. The semigroups $N_k(C_3)$

Consider the cyclic group $C_3 = \langle a \rangle = \{e, a, a^2 : a^3 = e\}$ generated by $a = e^{2\pi i/3} \in \mathbb{C}$.

Let us introduce the notations

$$\begin{aligned} |^x_y &= |^y_x = \langle \{x, y\} \rangle, \\ \vee_x &= \{F \subset C_3 : |F| \geq 2, x \in F\}, \\ \Delta &= \{F \subset C_3 : |F| \geq 2\}, \\ \bigcirc &= \{C_3\}. \end{aligned}$$

In these notations, $N_k(C_3) = \{e, a, a^3, |_{e}^a, |_{e}^{a^2}, |_{a}^{a^2}, \vee_e, \vee_a, \vee_{a^2}, \circ\}$ for $k \geq 3$ and $N_2(C_3) = N_k(C_3) \cup \{\Delta\}$.

*	a	a^2	e	$ _{e}^{a^2}$	$ _{e}^a$	$ _{a}^{a^2}$	\vee_{a^2}	\vee_a	\vee_e	\circ	Δ
a	a^2	e	a	$ _{e}^{a^2}$	$ _{e}^{a^2}$	$ _{e}^a$	\vee_e	\vee_{a^2}	\vee_a	\circ	Δ
a^2	e	a	a^2	$ _{e}^a$	$ _{e}^{a^2}$	$ _{a}^{a^2}$	\vee_a	\vee_e	\vee_{a^2}	\circ	Δ
e	a	a^2	e	$ _{a}^{a^2}$	$ _{e}^a$	$ _{e}^{a^2}$	\vee_{a^2}	\vee_a	\vee_e	\circ	Δ
$ _{e}^{a^2}$	$ _{e}^{a^2}$	$ _{e}^a$	$ _{a}^{a^2}$	\circ	\circ	\circ	$ _{e}^a$	$ _{e}^{a^2}$	$ _{a}^{a^2}$	\circ	Δ
$ _{e}^a$	$ _{a}^{a^2}$	$ _{e}^{a^2}$	$ _{e}^a$	\circ	\circ	\circ	$ _{e}^{a^2}$	$ _{a}^{a^2}$	$ _{e}^a$	\circ	Δ
$ _{a}^{a^2}$	$ _{e}^a$	$ _{e}^{a^2}$	$ _{e}^{a^2}$	\circ	\circ	\circ	$ _{a}^{a^2}$	$ _{e}^a$	$ _{e}^{a^2}$	\circ	Δ
\vee_{a^2}	\vee_e	\vee_a	\vee_{a^2}	\circ	\circ	\circ	\vee_a	\vee_e	\vee_{a^2}	\circ	Δ
\vee_a	\vee_{a^2}	\vee_e	\vee_a	\circ	\circ	\circ	\vee_e	\vee_{a^2}	\vee_a	\circ	Δ
\vee_e	\vee_a	\vee_{a^2}	\vee_e	\circ	\circ	\circ	\vee_{a^2}	\vee_a	\vee_e	\circ	Δ
\circ	\circ	\circ	\circ	\circ	\circ	\circ	\circ	\circ	\circ	\circ	Δ
Δ	Δ	Δ	Δ	\circ	\circ	\circ	Δ	Δ	Δ	\circ	Δ

Table 2: The Cayley table for the semigroups $N_k(C_3)$

Analyzing the Cayley Table 2 for the semigroups $N_k(C_3)$, we can establish the following properties.

The semigroup $N_2(C_3)$ contains 11 elements. Among them, there are 4 idempotents: e, \vee_e, \circ, Δ . Two idempotents are right zeros. The orbit semigroup $N_2(C_3)/C_3$ contains 5 elements. The semigroup $N_2(C_3)$ contains a transversal semigroup $T(N_2(C_3)) = \{e, |_{e}^a, \vee_e, \circ, \Delta\}$.

For $k \geq 3$, the semigroups $N_k(C_3)$ contain 10 elements. Among them, there are 3 idempotents e, \vee_e, \circ , which commute. The set $E(N_k(C_3))$ of idempotents of $N_k(C_3)$ is isomorphic to the semilattice $3 = \{0, 1, 2\}$ endowed with the operation of minimum. The orbit semigroups $N_k(C_3)/C_3$ contain 4 elements. The semigroups $N_k(C_3)$ contain transversal semigroups $T(N_k(C_3)) = \{e, |_{e}^a, \vee_e, \circ\}$.

Therefore, $N_k(C_3) = \{x, x \cdot |_{e}^a, x \cdot \vee_e, \circ \mid x \in C_3\}$ for $k \geq 3$ and $N_2(C_3) = N_k(C_3) \cup \{\Delta\}$.

We will prove that the automorphism groups $\text{Aut}(N_k(C_3))$ of the semigroups $N_k(C_3)$ are isomorphic to the holomorph $\text{Hol}(C_3)$ of the group C_3 .

We recall that the *holomorph* $\text{Hol}(G)$ of a group G (see [23]) is the semidirect product $G \rtimes \text{Aut}(G) := (G \times \text{Aut}(G), \star)$ of the group G with its automorphism group $\text{Aut}(G)$ endowed with the group operation

$$(x, f) \star (y, g) = (x \cdot f(y), f \circ g).$$

It is known¹ that, for the cyclic group C_3 , its holomorph $\text{Hol}(C_3)$ is isomorphic to the symmetric group S_3 .

Proposition 2.3. *For each $k \in \mathbb{N} \setminus \{1\}$, the automorphism group $\text{Aut}(N_k(C_3))$ is isomorphic to the holomorph $\text{Hol}(C_3)$ of the cyclic group C_3 and, hence, is isomorphic to the symmetric group S_3 .*

Proof. Let $\psi : N_k(C_3) \rightarrow N_k(C_3)$ be an automorphism. Then the restriction of ψ to C_3 is an automorphism of C_3 by Proposition 2.1. Hence, $\psi(e) = e$.

¹https://groupprops.subwiki.org/wiki/Holomorph_of_a_group

Since the semigroup $N_2(C_3)$ contains two right zeros, and the set of right zeros is preserved by automorphisms of semigroups, $\psi(\{\Delta, \bigcirc\}) = \{\Delta, \bigcirc\}$ for any automorphism $\psi : N_2(C_3) \rightarrow N_2(C_3)$. Assume that $\psi(\Delta) = \bigcirc$ and $\psi(\bigcirc) = \Delta$. Then $\psi(\Delta \cdot |_e^a) = \psi(\bigcirc) = \Delta$, but $\psi(\Delta) \cdot \psi(|_e^a) = \bigcirc \cdot \psi(|_e^a) \in \bigcirc \cdot (N_2(C_3) \setminus \{\Delta, \bigcirc\}) = \{\bigcirc\}$. So, we arrive to a contradiction with $\psi \in \text{Aut}(N_2(C_3))$. Therefore, $\psi(\Delta) = \Delta$ and $\psi(\bigcirc) = \bigcirc$ for each automorphism $\psi : N_2(C_3) \rightarrow N_2(C_3)$.

The k -linked upfamily \bigcirc is the zero of the semigroups $N_k(C_3)$ for $k \geq 3$. Since the zero is preserved by the automorphisms of semigroups, $\psi(\bigcirc) = \bigcirc$ for any automorphism $\psi : N_k(C_3) \rightarrow N_k(C_3)$, $k \geq 3$.

Taking into account that $\psi(E(N_k(C_3))) = E(N_k(C_3))$, we conclude that $\psi(\vee_e) = \vee_e$ for each $\psi \in \text{Aut}(N_k(C_3))$, $k \geq 2$. Therefore, $\psi(x \cdot \vee_e) = \psi(x) \cdot \psi(\vee_e) = \psi(x) \cdot \vee_e$ for any $x \in C_3$.

Let $\psi(|_e^a) = c \cdot |_e^a$ for some $c \in C_3$. Then $\psi(x \cdot |_e^a) = \psi(x) \cdot c \cdot |_e^a$ for any $x \in C_3$. It follows that an element c can be chosen from C_3 in one of three ways. Hence, analyzing the Cayley Table 2, we can check that each automorphism of C_3 can be extended to an automorphism of $N_k(C_3)$ exactly in three different ways.

For any pair $(c, f) \in C_3 \times \text{Aut}(C_3)$, we now consider the automorphism $\psi_{(c,f)}$ of $N_k(C_3)$ defined by

$$\begin{aligned} \psi_{c,f}(x) &= f(x), \quad \psi_{c,f}(x \cdot \vee_e) = f(x) \cdot \vee_e, \quad \psi_{c,f}(x \cdot |_e^a) = f(x) \cdot c \cdot |_e^a \\ &\text{for } x \in C_3, \quad \psi_{c,f}(\bigcirc) = \bigcirc, \end{aligned}$$

and $\psi_{c,f}(\Delta) = \Delta$ for the semigroup $N_2(C_3)$.

It follows that each automorphism of $N_k(C_3)$ is of the form $\psi_{c,f}$ for some $(c, f) \in C_3 \times \text{Aut}(C_3)$.

Observe that, for any $(b, f), (c, g) \in C_3 \times \text{Aut}(C_3)$ and $x \in C_3$, we get

$$\begin{aligned} \psi_{b,f} \circ \psi_{c,g}(x) &= \psi_{b,f}(g(x)) = f \circ g(x), \\ \psi_{b,f} \circ \psi_{c,g}(x \cdot \vee_e) &= \psi_{b,f}(g(x) \cdot \vee_e) = f \circ g(x) \cdot \vee_e, \\ \psi_{b,f} \circ \psi_{c,g}(x \cdot |_e^a) &= \psi_{b,f}(g(x) \cdot c \cdot |_e^a) = f \circ g(x) \cdot f(c) \cdot |_e^a, \\ \psi_{b,f} \circ \psi_{c,g}(\bigcirc) &= \psi_{b,f}(\bigcirc) = \bigcirc, \\ \psi_{b,f} \circ \psi_{c,g}(\Delta) &= \psi_{b,f}(\Delta) = \Delta. \end{aligned}$$

Consequently, $\psi_{b,f} \circ \psi_{c,g} = \psi_{b,f(c), f \circ g}$. Hence, for each $k \in \mathbb{N} \setminus \{1\}$, the group $\text{Aut}(N_k(C_3))$ is isomorphic to the holomorph $\text{Hol}(C_3) = C_3 \rtimes \text{Aut}(C_3) \cong C_3 \rtimes C_2$ of the group C_3 , which is known to be isomorphic to the symmetric group S_3 . \square

3. The automorphism groups of the extensions of finite monogenic semigroups

A semigroup $\langle a \rangle = \{a^n\}_{n \in \mathbb{N}}$ generated by a single element a is called *monogenic* or *cyclic*. If a monogenic semigroup is infinite, then it is isomorphic to the additive semigroup \mathbb{N} . A finite monogenic semigroup $S = \langle a \rangle$ has also a simple structure, see [21]. There are positive integer numbers r and m called the *index* and the *period* of S such that

- $S = \{a, a^2, \dots, a^{r+m-1}\}$ and $r + m - 1 = |S|$;
- $a^{r+m} = a^r$;
- $\{a^r, a^{r+1}, \dots, a^{r+m-1}\}$ is the cyclic maximal subgroup of S with the identity element $e = a^r$ and the generator a^{n+1} , where $n \in m \cdot \mathbb{N} \cap \{r, \dots, r + m - 1\}$.

By $M_{r,m}$, we denote a finite monogenic semigroup of index r and period m , and the maximal subgroup of $M_{r,m}$ is denoted by C_m . Note that $|M_{r,m}| = r + m - 1$.

A homomorphism $\varphi : S \rightarrow I$ from a semigroup S into an ideal $I \subset S$ is called a *homomorphic retraction*, if $\varphi(a) = a$ for any element $a \in I$.

Let e be the identity element of the maximal subgroup C_m of a monogenic semigroup $M_{r,m}$. The following lemma was proved in [15, Lemma 1.3].

Lemma 3.1. *The map $\varphi : M_{r,m} \rightarrow C_m$, $\varphi(x) = ex$, is a homomorphic retraction, and $\varphi(x)y = xy$ for any $x \in M_{r,m}$ and $y \in C_m$.*

Proposition 3.2. *Let $M_{r,m}$ be a monogenic semigroup of index $r \geq 3$. If $\psi : N_k(M_{r,m}) \rightarrow N_k(M_{r,m})$ is an automorphism, then $\psi(s) = s$ for any $s \in M_{r,m}$.*

Proof. Let $M_{r,m} = \langle a \rangle = \{a, \dots, a^r, \dots, a^{r+m-1}\}$. We assume that $\psi(a) = \mathcal{A} \in N_k(M_{r,m}) \setminus \{a\}$. Since ψ is an automorphism of $N_k(M_{r,m})$, $\psi(a * N_k(M_{r,m})) = \psi(a) * \psi(N_k(M_{r,m})) = \psi(a) * N_k(M_{r,m})$. Hence, the semigroups $a * N_k(M_{r,m})$ and $\mathcal{A} * N_k(M_{r,m})$ are isomorphic. It is easy to see that $\mathcal{A} * N_k(M_{r,m}) \subset a * N_k(M_{r,m})$. Taking into account that, in the extensions $N_k(M_{r,m})$ of a monogenic semigroup of index $r \geq 3$, the equality $\mathcal{L} * \mathcal{M} = a^2$ yields $\mathcal{L} = \mathcal{M} = a$, we conclude that $a^2 \in a * N_k(M_{r,m}) \setminus \mathcal{A} * N_k(M_{r,m})$ and, hence, $|\mathcal{A} * N_k(M_{r,m})| < |a * N_k(M_{r,m})|$. This contradiction proves that $\psi(a) = a$ and, therefore, $\psi(a^i) = (\psi(a))^i = a^i$ for any $i \in \{2, \dots, r + m - 1\}$. \square

Proposition 3.3. *If $\psi : N_k(M_{1,m}) \rightarrow N_k(M_{1,m})$ is an automorphism, then $\psi(M_{1,m}) = M_{1,m}$.*

Proof. Since a monogenic semigroup $M_{1,m}$ is isomorphic to the cyclic group C_m , we conclude that $\psi(M_{1,m}) = M_{1,m}$ according to Proposition 2.1. \square

The following theorem shows that there exist automorphisms of the semigroups $N_k(S)$ that are not extensions of automorphisms of a semigroup S .

Consider the monogenic semigroup $M_{2,m} = \langle a \rangle = \{a, \dots, a^{m+1} \mid a^{m+2} = a^2\}$ and let

$$\mathbb{X} = \{\mathcal{M} \in N_k(M_{2,m}) \mid \{a, a^{m+1}\} \in \mathcal{M}\} \setminus \{a^{m+1}\}.$$

Theorem 3.4. *A homomorphism $\psi : M_{2,m} \rightarrow N_k(M_{2,m})$, $\psi(a) = \mathcal{A}$, can be extended to an automorphism $\psi : N_k(M_{2,m}) \rightarrow N_k(M_{2,m})$ if and only if $\mathcal{A} \in \mathbb{X}$. The automorphism group $\text{Aut}(N_k(M_{2,m}))$ contains the symmetric group $S_{\mathbb{X}}$ as a subgroup.*

Proof. Let $\mathcal{A} \in \mathbb{X}$. Since, for $m \geq 2$, a^m is the identity of the cyclic group $C_m = \{a^2, \dots, a^{m+1}\}$, and $ay \in C_m$, we have $ay = \varphi(ay) = a^m(ay) = a^{m+1}y$ for any $y \in M_{2,m}$ according to Lemma 3.1. The monogenic semigroup $M_{2,1} = \{a, a^2\}$ is a null semigroup with the zero a^2 , and, hence, $ay = a^2 = a^2y$ for any $y \in M_{2,1}$. This implies that $a * \mathcal{M} = \mathcal{A} * \mathcal{M}$ and $\mathcal{M} * a = \mathcal{M} * \mathcal{A}$ for any $\mathcal{M} \in N_k(M_{2,m})$, $m \in \mathbb{N}$. Indeed, $aM = \{a, a^{m+1}\}M \in \mathcal{A} * \mathcal{M}$ and $Ma = M\{a, a^{m+1}\} \in \mathcal{M} * \mathcal{A}$ for any $M \in \mathcal{M}$. Hence, $a * \mathcal{M} \subset \mathcal{A} * \mathcal{M}$ and $\mathcal{M} * a \subset \mathcal{M} * \mathcal{A}$. Since $\mathcal{A} \ni \{a, a^{m+1}\}$ is k -linked, $a \in A$ or $a^{m+1} \in A$ for any $A \in \mathcal{A}$. Taking into account that $\bigcup_{s \in A} sM_s \supset aM_a \in a * \mathcal{M}$ or $\bigcup_{s \in A} sM_s \supset a^{m+1}M_{a^{m+1}} = aM_{a^{m+1}} \in a * \mathcal{M}$ for any basic set $\bigcup_{s \in A} sM_s \in \mathcal{A} * \mathcal{M}$, we conclude that $\mathcal{A} * \mathcal{M} \subset a * \mathcal{M}$. Let $\bigcup_{s \in M} sA_s \in \mathcal{M} * \mathcal{A}$. Since $\bigcup_{s \in M} sA_s \supset \bigcup_{s \in M} (s\{a\} = s\{a^{m+1}\}) = Ma$, $\mathcal{M} * \mathcal{A} \subset \mathcal{M} * a$. Therefore, $\mathcal{M} * \mathcal{A} = \mathcal{M} * a$ and $\mathcal{A} * \mathcal{M} = a * \mathcal{M}$.

Note that $\{a^2\} = \{a, a^{m+1}\}\{a, a^{m+1}\} \in \mathcal{A} * \mathcal{A}$. Then the linkedness of $\mathcal{A} * \mathcal{A}$ implies that $\mathcal{A} * \mathcal{A} = a^2$. By the same arguments, $a * \mathcal{A} = \mathcal{A} * a = a^2$. Therefore, $\psi(a^i) = \mathcal{A}^i = a^i$ for any $i \geq 2$.

Let us put $\psi(\mathcal{A}) = a$ and $\psi(\mathcal{M}) = \mathcal{M}$ for any $\mathcal{M} \in N_k(M_{2,m}) \setminus \{a, \mathcal{A}\}$. Then the above-proved equalities imply that $\mathcal{M} * \mathcal{L} = \psi(\mathcal{M}) * \psi(\mathcal{L})$ for any $\mathcal{L}, \mathcal{M} \in N_k(M_{2,m})$. Since $\mathcal{M} * \mathcal{L} \in N_k(M_{2,m}) \setminus \{a, \mathcal{A}\}$ for any $\mathcal{L}, \mathcal{M} \in N_k(M_{2,m})$, we have $\psi(\mathcal{M} * \mathcal{L}) = \mathcal{M} * \mathcal{L} = \psi(\mathcal{M}) * \psi(\mathcal{L})$. Hence, ψ is an automorphism of $M_{2,m}$.

Let $\mathcal{A} \notin \mathbb{X}$. If $\psi(a) = a^i$ for some $i \in \{2, \dots, m+1\}$, then $\psi(M_{2,m}) = M_{2,m} \setminus \{a\}$. Thus, ψ is not one-to-one. Therefore, $\mathcal{A} \notin \mathbb{X} \cup M_{2,m}$, and, thus, $M_{2,m} \subset a * N_k(M_{2,m}) \setminus \mathcal{A} * N_k(M_{2,m})$. Consequently, $|a * N_k(M_{2,m})| > |\mathcal{A} * N_k(M_{2,m})|$. Hence, ψ cannot be a bijection.

Let us prove that the automorphism group $\text{Aut}(N_k(M_{2,m}))$ contains the symmetric group $S_{\mathbb{X}}$ as a subgroup. Let us extend any bijection ψ of a set \mathbb{X} to $N_k(M_{2,m})$, by putting $\psi(\mathcal{L}) = \mathcal{L}$ for any $\mathcal{L} \in N_k(M_{2,m}) \setminus \mathbb{X}$. As we have shown above, $\mathcal{M} * \mathcal{A} = \mathcal{M} * a = \mathcal{M} * \mathcal{B}$ and $\mathcal{A} * \mathcal{M} = a * \mathcal{M} = \mathcal{B} * \mathcal{M}$ for any $\mathcal{A}, \mathcal{B} \in \mathbb{X}, \mathcal{M} \in N_k(M_{2,m})$. In addition, $\mathcal{A} * \mathcal{B} \ni \{a, a^{m+1}\} \{a, a^{m+1}\} = \{a^2\}$. So, the linkedness of $\mathcal{A} * \mathcal{B}$ implies that $\mathcal{A} * \mathcal{B} = a^2$ for any $\mathcal{A}, \mathcal{B} \in \mathbb{X}$. Taking into account that $\mathcal{M} * \mathcal{L} \in N_k(M_{2,m}) \setminus \{a, \mathcal{A}\}$ for any $\mathcal{L}, \mathcal{M} \in N_k(M_{2,m})$, we conclude that $\psi(\mathcal{M} * \mathcal{L}) = \mathcal{M} * \mathcal{L}$. Therefore, $\psi(\mathcal{M} * \mathcal{L}) = \mathcal{M} * \mathcal{L} = \psi(\mathcal{M}) * \psi(\mathcal{L})$ for any $\mathcal{L}, \mathcal{M} \in N_k(M_{2,m})$. Hence, ψ is an automorphism of $N_k(M_{2,m})$. \square

Now, we describe the structure of the automorphism groups of semigroups of k -linked upfamilies on monogenic semigroups $M_{r,m}$ of order $|M_{r,m}| \leq 3$.

It is well known that $\text{Aut}(M_{r,m}) \cong C_1$ for $r \geq 2$ and $\text{Aut}(M_{1,m}) \cong \text{Aut}(C_m) \cong C_{\varphi(m)}$, where $\varphi(m)$ is the value of Euler's function for $m \in \mathbb{N}$.

3.1. The semigroups $N_k(M_{1,1})$, $N_k(M_{1,2})$, and $N_k(M_{2,1})$

For the trivial monogenic semigroup $M_{1,1}$, the semigroups $N_k(M_{1,1})$ are trivial as well. Therefore, $\text{Aut}(N_k(M_{1,1})) \cong \text{Aut}(M_{1,1}) \cong \text{Aut}(C_1) \cong C_1$.

For a semigroup $M_{r,m} = \langle a \rangle$ with $m+r=3$, the semigroups $N_k(M_{r,m})$ contain the two principal ultrafilters a and a^2 and the k -linked upfamily $\{M_{r,m}\}$.

Taking into account that $M_{1,2}$ is isomorphic to C_2 , we conclude that

$$\text{Aut}(N_k(M_{1,2})) \cong \text{Aut}(N_k(C_2)) \cong C_1.$$

Consider the semigroup $N_k(M_{2,1})$. The proof of Theorem 3.4 implies that $\psi(a^2) = a^2$ for any $\psi \in \text{Aut}(N_k(M_{2,1}))$. Then, except for the identity automorphism, the group $\text{Aut}(N_k(M_{2,1}))$ contains the automorphism ψ with $\psi(a) = \{M_{2,1}\}$, $\psi(\{M_{2,1}\}) = a$ according to Theorem 3.4. Consequently, $\text{Aut}(N_k(M_{2,1})) \cong C_2$.

3.2. The semigroups $N_k(M_{1,3})$

The semigroup $M_{1,3}$ is isomorphic to the cyclic group C_3 . Therefore,

$$\text{Aut}(N_k(M_{1,3})) \cong \text{Aut}(N_k(C_3)) \cong S_3.$$

3.3. The semigroups $N_k(M_{2,2})$

Consider the semigroup $M_{2,2} = \{a, a^2, a^3 \mid a^4 = a^2\}$. The semigroup $N_2(M_{2,2})$ contains 11 elements, while the semigroups $N_k(M_{2,2})$ for $k \geq 3$ have 10 elements.

In the Cayley Table 3 for $N_k(M_{2,2})$, we denoted, by a^i , the principal ultrafilter generated by $\{a^i\}$ and introduced the notations

$$\begin{matrix} |x \\ |y \end{matrix} = \begin{matrix} |x \\ |y \end{matrix} = \langle \{a^x, a^y\} \rangle, \quad \forall x = \{F \subset M_{2,2} : |F| \geq 2, a^x \in F\}, \quad \bigcirc = \{M_{2,2}\}.$$

In these notations,

$$N_k(M_{2,2}) = \{a, a^2, a^3, |1^2, |1^3, |2^3, \forall_1, \forall_2, \forall_3, \bigcirc\}$$

for $k \geq 3$ and $N_2(M_{2,2}) = N_k(M_{2,2}) \cup \{\Delta\}$.

*	a	a^2	a^3	$ \frac{3}{1}$	$ \frac{3}{1}$	$ \frac{3}{2}$	\vee_1	\vee_2	\vee_3	\bigcirc	Δ
a	a^2	a^3	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2
a^2	a^3	a^2	a^3	$ \frac{3}{2}$	a^3	$ \frac{3}{2}$	a^3	$ \frac{3}{2}$	a^3	$ \frac{3}{2}$	a^3
a^3	a^2	a^3	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2
$ \frac{3}{1}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$
$ \frac{3}{1}$	a^2	a^3	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2
$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$
\vee_1	a^2	a^3	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2
\vee_2	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$
\vee_3	a^2	a^3	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2
\bigcirc	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$	$ \frac{3}{2}$
Δ	a^2	a^3	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2	$ \frac{3}{2}$	a^2

Table 3: The Cayley table for the semigroups $N_k(M_{2,2})$

Let $\psi \in \text{Aut}(N_k(M_{2,2}))$. Then $\psi(a^i) = a^i$ for $i \in \{2, 3\}$, according to proof of Theorem 3.4. Since $|\frac{3}{2}$ is the unique idempotent in $N_k(M_{2,2}) \setminus M_{2,2}$, we conclude that $\psi(|\frac{3}{2}) = |\frac{3}{2}$.

Consider the semigroups $N_k(M_{2,2}) = \{a, a^2, a^3, |\frac{3}{1}, |\frac{3}{1}, |\frac{3}{2}, \vee_1, \vee_2, \vee_3, \bigcirc\}$ for $k \geq 3$. Let $\mathbb{X} = \{a, |\frac{3}{1}, \vee_1, \vee_3\}$, $\mathbb{Y} = \{|\frac{3}{1}, \vee_2, \bigcirc\}$, and $x \in \mathbb{X}$, $y \in \mathbb{Y}$. Assume that $\psi(y) = x \in \mathbb{X}$. Then $\psi(y * y) = \psi(|\frac{3}{2}) = |\frac{3}{2}$ and $\psi(y) * \psi(y) = x * x = a^2$, which contradicts the assertion that ψ is an automorphism. Therefore, $\psi(x) \in \mathbb{X}$ and $\psi(y) \in \mathbb{Y}$ for any $x \in \mathbb{X}$, $y \in \mathbb{Y}$.

Analyzing the Cayley Table 3 for the semigroups $N_k(M_{2,2})$, we can establish that the semigroups $N_k(M_{2,2})$ are commutative, and $x_1 * s = x_2 * s$, $y_1 * s = y_2 * s$, $x_1 * y_1 = x_2 * y_2$, $x_1 * x_2 = a^2$, $y_1 * y_2 = |\frac{3}{2}$ for any $x_1, x_2 \in \mathbb{X}$, $y_1, y_2 \in \mathbb{Y}$, $s \in M_{2,2}$. Consequently, each permutation of \mathbb{X} and each permutation of \mathbb{Y} define the automorphism of the semigroup $N_k(M_{2,2})$. Therefore, $\text{Aut}(N_k(M_{2,2})) \cong S_{\mathbb{X}} \times S_{\mathbb{Y}} \cong S_4 \times S_3$ for $k \geq 3$.

Consider the semigroup $N_2(M_{2,2}) = N_k(M_{2,2}) \cup \{\Delta\}$. Let $\mathbb{X}' = \mathbb{X} \cup \{\Delta\}$. By the same arguments, $\text{Aut}(N_2(M_{2,2})) \cong S_{\mathbb{X}'} \times S_{\mathbb{Y}} \cong S_5 \times S_3$.

3.4. The semigroups $N_k(M_{3,1})$

Consider the semigroup $M_{3,1} = \{a, a^2, a^3 \mid a^4 = a^3\}$. In Cayley Table 4 for the semigroups $N_k(M_{3,1})$, we use the similar notations as for the semigroups $N_k(M_{2,2})$.

Let $\psi \in \text{Aut}(N_k(M_{3,1}))$. Then $\psi(a^i) = a^i$ for $i \in \{1, 2, 3\}$, according to Proposition 3.2.

Consider the semigroups $N_k(M_{3,1}) = \{a, a^2, a^3, |\frac{2}{1}, |\frac{3}{1}, |\frac{3}{2}, \vee_1, \vee_2, \vee_3, \bigcirc\}$ for $k \geq 3$. Let $\mathbb{X} = \{|\frac{2}{1}, |\frac{3}{1}, \vee_1, \bigcirc\}$, and $\mathbb{Y} = \{\vee_2, \vee_3\}$.

We claim that $\psi(|\frac{3}{2}) = |\frac{3}{2}$. Assume that $\psi(|\frac{3}{2}) = x \in \mathbb{X}$. Then $\psi(a * |\frac{3}{2}) = \psi(a^3) = a^3$ and $\psi(a) * \psi(|\frac{3}{2}) = a * x = |\frac{3}{2}$, and we arrive to a contradiction with $\psi \in \text{Aut}(N_k(M_{3,1}))$. In the same way, by assuming that $\psi(|\frac{3}{2}) = y \in \mathbb{Y}$, we have $\psi(a * |\frac{2}{1}) = \psi(|\frac{3}{2}) = y$, but $\psi(a) * \psi(|\frac{2}{1}) = a * \psi(|\frac{2}{1}) \notin \mathbb{Y}$. These contradictions show that $\psi(|\frac{3}{2}) = |\frac{3}{2}$.

Then, in the same way as for the semigroups $N_k(M_{2,2})$, we establish that $\text{Aut}(N_k(M_{3,1})) \cong S_{\mathbb{X}} \times S_{\mathbb{Y}} \cong S_4 \times S_2$ for $k \geq 3$.

*	a	a^2	a^3	$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	\vee_1	\vee_2	\vee_3	\bigcirc	\triangle
a	a^2	a^3	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3
a^2	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3
$\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3
$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
\vee_1	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3
\vee_2	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
\vee_3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
\bigcirc	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3	a^3	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	a^3
\triangle	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3

Table 4: The Cayley table for the semigroups $N_k(M_{3,1})$

Consider the semigroup $N_2(M_{3,1}) = N_k(M_{3,1}) \cup \{\Delta\}$. Let $\mathbb{Y}' = \mathbb{Y} \cup \{\Delta\}$. By the same arguments, $\text{Aut}(N_2(M_{3,1})) \cong S_{\mathbb{X}} \times S_{\mathbb{Y}'} \cong S_4 \times S_3$.

4. The automorphism groups of the semigroups $N_k(O_X)$, $N_k(LO_X)$, $N_k(RO_X)$, $N_k(AO_X)$ and $N_k((O_X)^{+0})$

A semigroup S is said to be a *left (right) zero semigroup*, if $ab = a$ ($ab = b$) for any $a, b \in S$. By LO_X and RO_X , we denote the left zero semigroup and the right zero semigroup on a set X , respectively. If X is finite and has cardinality $|X| = n$, then, instead of LO_X and RO_X , we use LO_n and RO_n , respectively.

Proposition 4.1. *If S is a left (right) zero semigroup, then, for each $k \in \mathbb{N} \setminus \{1\}$, the extension $N_k(S)$ is a left (right) zero semigroup as well.*

Proof. Let S be a left zero semigroup. Then

$$\mathcal{L} * \mathcal{M} = \left\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L}, \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle = \left\langle \bigcup_{a \in L} \{a\} : L \in \mathcal{L} \right\rangle = \mathcal{L}$$

for any $\mathcal{L}, \mathcal{M} \in N_k(S)$. Therefore, $N_k(S)$ is a left zero semigroup as well.

For a right zero semigroup, the proof is similar. □

Proposition 4.2. *If X is a left zero semigroup or a right zero semigroup, then, for each $k \in \mathbb{N} \setminus \{1\}$, $\text{Aut}(N_k(X))$ is isomorphic to the symmetric group $S_{N_k(X)}$.*

Proof. In Proposition 4.1, it was shown that the extensions $N_k(S)$ of a left (right) zero semigroup S are left (right) zero semigroups as well. Each permutation on a left (right) zero semigroup is an automorphism. Indeed, $\psi(x * y) = \psi(x) = \psi(x) * \psi(y)$ and $\psi(x * y) = \psi(y) = \psi(x) * \psi(y)$ for any elements x and y of the left zero semigroup and the right zero semigroup, respectively. Therefore, $\text{Aut}(N_k(X)) \cong S_{N_k(X)}$. □

Using the results of Table 1 and Proposition 4.2 in Table 5, we present the automorphism groups of the semigroups $N_k(LO_n)$ and $N_k(RO_n)$ for $k \in \{2, 3, 4\}$ and $n \leq 5$.

n	$\text{Aut}(N_2(LO_n))$	$\text{Aut}(N_3(LO_n))$	$\text{Aut}(N_4(LO_n))$
1	C_1	C_1	C_1
2	S_3	S_3	S_3
3	S_{11}	S_{10}	S_{10}
4	S_{80}	S_{54}	S_{53}
5	S_{2645}	S_{762}	S_{687}

Table 5: The automorphism groups of the semigroups $N_k(LO_n)$ for $k \in \{2, 3, 4\}$ and $n \leq 5$

A semigroup S is called a *null semigroup*, if there exists an element $z \in S$ such that $xy = z$ for any $x, y \in S$. In this case, the element z is the zero of S . All null semigroups on the same set are isomorphic. By O_X , we denote a null semigroup on a set X . If X is finite and has cardinality $|X| = n$, then, instead of O_X , we use O_n .

Proposition 4.3. *If S is a null semigroup, then, for each $k \in \mathbb{N} \setminus \{1\}$, the extension $N_k(S)$ is a null semigroup as well.*

Proof. Let S be a null semigroup. Then there exists $z \in S$ such that $xy = z$ for all $x, y \in S$. Therefore,

$$\mathcal{L} * \mathcal{M} = \left\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L}, \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle = \left\langle \bigcup_{a \in L} \{z\} : L \in \mathcal{L}, \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle = z$$

for any $\mathcal{L}, \mathcal{M} \in N_k(S)$. Consequently, $N_k(S)$ is a null semigroup with the zero $z \in S \subset N_k(S)$. \square

Since $N_k(O_1) \cong O_1$, the automorphism groups of the semigroups $N_k(O_1)$ are trivial. In the following proposition, we describe the automorphism group of the semigroups $N_k(O_X)$ on a set X of cardinality $|X| \geq 2$.

Proposition 4.4. *Let z be the zero of the null semigroup O_X on a set X of cardinality $|X| \geq 2$. For each $k \in \mathbb{N} \setminus \{1\}$, the automorphism group of the semigroup $N_k(O_X)$ is isomorphic to the symmetric group $S_{N_k(O_X) \setminus \{z\}}$.*

Proof. In Proposition 4.3, it was proved that the semigroups $N_k(O_X)$ are null semigroups with the zero z . Taking into account that z is the zero of the semigroups $N_k(O_X)$, we conclude that $\psi(z) = z$ for any $\psi \in \text{Aut}(N_k(O_X))$. Each permutation on the set $N_k(O_X) \setminus \{z\}$ defines an automorphism. Indeed, $\psi(x * y) = z = \psi(x) * \psi(y)$ for any elements $x, y \in N_k(O_X)$. Therefore, $\text{Aut}(N_k(O_X)) \cong S_{N_k(O_X) \setminus \{z\}}$. \square

Using the results of Table 1 and Proposition 4.4 in Table 6, we present the automorphism groups of the semigroups $N_k(O_n)$ for $k \in \{2, 3, 4\}$ and $n \leq 5$.

A semigroup S is said to be an *almost null semigroup*, if there exist the distinct elements $a, z \in S$ such that $aa = a$ and $xy = z$ for any $(x, y) \in S \times S \setminus \{(a, a)\}$. In this case, the element z is the zero of S , and a is the unique idempotent in $S \setminus \{z\}$. All almost null semigroups on the same set are isomorphic. By AO_X , we denote an almost null semigroup on a set X . If X is finite and has cardinality $|X| = n$, then we use AO_n instead of AO_X .

n	$\text{Aut}(N_2(O_n))$	$\text{Aut}(N_3(O_n))$	$\text{Aut}(N_4(O_n))$
1	C_1	C_1	C_1
2	C_2	C_2	C_2
3	S_{10}	S_9	S_9
4	S_{79}	S_{53}	S_{52}
5	S_{2644}	S_{761}	S_{686}

Table 6: The automorphism groups of the semigroups $N_k(O_n)$ for $k \in \{2, 3, 4\}$ and $n \leq 5$

It is easy to check that the automorphism groups of the semigroups $N_k(AO_2)$ are trivial. In the following theorem, we describe the automorphism groups of the semigroups $N_k(AO_X)$ on a set X of cardinality $|X| \geq 3$.

Theorem 4.5. *Let z be the zero of the almost null semigroup AO_X on a set X of cardinality $|X| \geq 3$, $\mathbb{A} = \{\mathcal{L} \in N_k(OA_X) \mid X \setminus \{a\} \in \mathcal{L}\}$, $\mathbb{B} = N_k(OA_X) \setminus \mathbb{A} = \{\mathcal{L} \in N_k(OA_X) \mid a \in L \text{ for any } L \in \mathcal{L}\}$, where a is the idempotent in $OA_X \setminus \{z\}$. For each $k \in \mathbb{N} \setminus \{1\}$, the automorphism group of the semigroup $N_k(AO_X)$ is isomorphic to the group $S_{\mathbb{A} \setminus \{z\}} \times S_{\mathbb{B} \setminus \{a, |z_a^z\}}$.*

Proof. Let $\mathcal{A} \in \mathbb{A}$, $\mathcal{L} \in N_k(OA_X)$. Then $\mathcal{A} * \mathcal{L} \ni (X \setminus \{a\}) \cdot L = \{z\} = L \cdot (X \setminus \{a\}) \in \mathcal{L} * \mathcal{A}$ for any $L \in \mathcal{L}$. Hence, the linkedness of $\mathcal{A} * \mathcal{L}$ and $\mathcal{L} * \mathcal{A}$ implies that $\mathcal{A} * \mathcal{L} = \mathcal{L} * \mathcal{A} = z$ for any $\mathcal{A} \in \mathbb{A}$, $\mathcal{L} \in N_k(OA_X)$. Consider any $(\mathcal{B}_1, \mathcal{B}_2) \in \mathbb{B} \times \mathbb{B} \setminus \{(a, a)\}$. Taking into account that $a \in B_1 \cap B_2$ and $|B_1| \geq 2$ or $|B_2| \geq 2$ for any $B_1 \in \mathcal{B}_1$, $B_2 \in \mathcal{B}_2$, we conclude that $\mathcal{B}_1 * \mathcal{B}_2 = \left\langle \bigcup_{x \in B_1} x * B_x : B_1 \in \mathcal{B}_1, \{B_x\}_{x \in B_1} \subset \mathcal{B}_2 \right\rangle = |z_a^z$.

Consequently, the semigroups $N_k(AO_X)$ contain three idempotents: a , z , and $|z_a^z$. Taking into account that z is the zero of $N_k(OA_X)$, and the set of idempotents of a semigroup is preserved by automorphisms, we conclude that $\psi(z) = z$. Hence, $\psi(\{a, |z_a^z\}) = \{a, |z_a^z\}$ for any $\psi \in \text{Aut}(N_k(AO_X))$. Since $\psi(|z_0^z) = \psi(z * |z_0^z) = \psi(z) * \psi(|z_0^z) \in \{z * |z_0^z, |z_0^z * z\} = \{|z_0^z\}$, we conclude that $\psi(|z_0^z) = |z_0^z$. Hence, $\psi(z) = z$ for any $\psi \in \text{Aut}(N_k((O_X)^{+0}))$.

Let us show that $\psi(\mathbb{A} \setminus \{z\}) = \mathbb{A} \setminus \{z\}$ and $\psi(\mathbb{B} \setminus \{a, |z_a^z\}) = \mathbb{B} \setminus \{a, |z_a^z\}$ for any $\psi \in \text{Aut}(N_k(AO_X))$. Assume that $\psi(\mathcal{B}) = \mathcal{A}$ for some $\mathcal{A} \in \mathbb{A} \setminus \{z\}$, $\mathcal{B} \in \mathbb{B} \setminus \{a, |z_a^z\}$. Then $\psi(\mathcal{B} * \mathcal{B}) = \psi(|z_a^z) = |z_a^z$, but $\psi(\mathcal{B}) * \psi(\mathcal{B}) = \mathcal{A} * \mathcal{A} = z$. This contradiction shows that $\psi(\mathcal{A}) \in \mathbb{A} \setminus \{z\}$, $\psi(\mathcal{B}) \in \mathbb{B} \setminus \{a, |z_a^z\}$ for any $\mathcal{A} \in \mathbb{A} \setminus \{z\}$, $\mathcal{B} \in \mathbb{B} \setminus \{a, |z_a^z\}$ and $\psi \in \text{Aut}(N_k(AO_X))$.

Each permutation on the set $\mathbb{A} \setminus \{z\}$ and each permutation on the set $\mathbb{B} \setminus \{a, |z_a^z\}$ define the automorphism $\psi : N_k(AO_X) \rightarrow N_k(AO_X)$. Indeed, $\psi(a * a) = \psi(a) = a = a * a = \psi(a) * \psi(a)$, $\psi(\mathcal{A} * \mathcal{L}) = \psi(z) = z = \psi(\mathcal{A}) * \psi(\mathcal{L})$, $\psi(\mathcal{L} * \mathcal{A}) = \psi(z) = z = \psi(\mathcal{L}) * \psi(\mathcal{A})$ for any $\mathcal{A} \in \mathbb{A}$, $\mathcal{L} \in N_k(OA_X)$, and $\psi(\mathcal{B}_1 * \mathcal{B}_2) = \psi(|z_a^z) = |z_a^z = \psi(\mathcal{B}_1) * \psi(\mathcal{B}_2)$ for any any $(\mathcal{B}_1, \mathcal{B}_2) \in \mathbb{B} \times \mathbb{B} \setminus \{(a, a)\}$.

Therefore, $\text{Aut}(N_k(OA_X)) \cong S_{\mathbb{A} \setminus \{z\}} \times S_{\mathbb{B} \setminus \{a, |z_a^z\}}$. \square

Let us note that, for a subsemigroup T of a semigroup S , the map $i : N_k(T) \rightarrow N_k(S)$, $i : \mathcal{A} \rightarrow \{L \subset S \mid L \supset A \in \mathcal{A}\}$, is an injective homomorphism. Thus, we can identify the semigroup $N_k(T)$ with the subsemigroup $i(N_k(T)) \subset N_k(S)$. Therefore, the set \mathbb{A} from Theorem 4.5 can be identified with the subsemigroup $N_k(X \setminus \{a\})$ of the semigroup $N_k(X)$. Consequently, for finite almost null semigroups, we have the following corollary.

Corollary 4.6. *For each $k \geq 2$ and $n \geq 3$, the automorphism group of the semigroup $N_k(OA_n)$ is isomorphic to the group $S_{|N_k(n-1)|-1} \times S_{|N_k(n)|-|N_k(n-1)|-3}$.*

Using the results of Table 1 and Corollary 4.6 in Table 7, we present the automorphism groups of the semigroups $N_k(AO_n)$ for $k \in \{2, 3, 4\}$ and $n \in \{2, 3, 4, 5\}$.

n	$\text{Aut}(N_2(AO_n))$	$\text{Aut}(N_3(AO_n))$	$\text{Aut}(N_4(AO_n))$
2	C_1	C_1	C_1
3	$C_2 \times S_5$	$C_2 \times S_4$	$C_2 \times S_4$
4	$S_{10} \times S_{66}$	$S_9 \times S_{41}$	$S_9 \times S_{40}$
5	$S_{79} \times S_{2562}$	$S_{53} \times S_{705}$	$S_{52} \times S_{631}$

Table 7: The automorphism groups of the semigroups $N_k(AO_n)$ for $k \in \{2, 3, 4\}$ and $n \in \{2, 3, 4, 5\}$

Let S be a semigroup, and let $0 \notin S$. The binary operation defined on S can be extended to $S \cup \{0\}$, by putting $0s = s0 = 0$ for all $s \in S \cup \{0\}$. The notation S^{+0} denotes a semigroup $S \cup \{0\}$ obtained from S by adjoining the extra zero 0 (regardless of whether S has or has not the zero).

Theorem 4.7. *Let z be the zero of the null semigroup O_X on a set X of cardinality $|X| \geq 2$, $\mathbb{A} = \{\mathcal{L} \in N_k((O_X)^{+0}) \mid X \in \mathcal{L}\}$, $\mathbb{B} = N_k((O_X)^{+0}) \setminus \mathbb{A} = \{\mathcal{L} \in N_k((O_X)^{+0}) \mid 0 \in L \text{ for any } L \in \mathcal{L}\}$, where 0 is the extra zero adjoined to O_X . For each $k \geq 2$, the automorphism group of the semigroup $N_k((O_X)^{+0})$ is isomorphic to the group $S_{\mathbb{A} \setminus \{z\}} \times S_{\mathbb{B} \setminus \{0, |z_0^z\}}$.*

Proof. It is easy to see that 0 is the zero of $N_k((O_X)^{+0})$.

Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{A}$. Then $\mathcal{A}_1 * \mathcal{A}_2 \ni X \cdot X = \{z\}$. The linkedness of $\mathcal{A}_1 * \mathcal{A}_2$ implies that $\mathcal{A}_1 * \mathcal{A}_2 = z$ for any $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{A}$. Consider any $\mathcal{B} \in \mathbb{B} \setminus \{0\}$, $\mathcal{L} \in N_k((O_X)^{+0}) \setminus \{0\}$. Taking into account that $0 \in B$ and $|B| \geq 2$, $|L| \geq 2$ for any $B \in \mathcal{B}$, $L \in \mathcal{L}$, we conclude that $\mathcal{B} * \mathcal{L} = \left\langle \bigcup_{x \in B} x * L_x : B \in \mathcal{B}, \{L_x\}_{x \in B} \subset \mathcal{L} \right\rangle = |z_0^z$ and $\mathcal{L} * \mathcal{B} = \left\langle \bigcup_{x \in L} x * B_x : L \in \mathcal{L}, \{B_x\}_{x \in L} \subset \mathcal{B} \right\rangle = |z_0^z$.

Consequently, the semigroup $N_k((O_X)^{+0})$ contains three idempotents: 0 , z , and $|z_0^z$. Taking into account that 0 is the zero of $N_k((O_X)^{+0})$, and the set of idempotents of a semigroup is preserved by automorphisms, we conclude that $\psi(0) = 0$. Hence, $\psi(\{z, |z_0^z\}) = \{z, |z_0^z\}$ for any $\psi \in \text{Aut}(N_k((O_X)^{+0}))$. Since $\psi(|z_0^z) = \psi(z * |z_0^z) = \psi(z) * \psi(|z_0^z) \in \{z * |z_0^z, |z_0^z * z\} = \{|z_0^z\}$, we conclude that $\psi(|z_0^z) = |z_0^z$. Hence, $\psi(z) = z$ for any $\psi \in \text{Aut}(N_k((O_X)^{+0}))$.

Let us show that $\psi(\mathbb{A} \setminus \{z\}) = \mathbb{A} \setminus \{z\}$ and $\psi(\mathbb{B} \setminus \{0, |z_0^z\}) = \mathbb{B} \setminus \{0, |z_0^z\}$ for any $\psi \in \text{Aut}(N_k((O_X)^{+0}))$. Assume that $\psi(\mathcal{A}) = \mathcal{B}$ for some $\mathcal{A} \in \mathbb{A} \setminus \{z\}$, $\mathcal{B} \in \mathbb{B} \setminus \{0, |z_0^z\}$. Then $\psi(\mathcal{A} * \mathcal{A}) = \psi(z) = z$, but $\psi(\mathcal{A}) * \psi(\mathcal{A}) = \mathcal{B} * \mathcal{B} = |z_0^z$. This contradiction shows that $\psi(\mathcal{A}) \in \mathbb{A} \setminus \{z\}$, $\psi(\mathcal{B}) \in \mathbb{B} \setminus \{0, |z_0^z\}$ for any $\mathcal{A} \in \mathbb{A} \setminus \{z\}$, $\mathcal{B} \in \mathbb{B} \setminus \{0, |z_0^z\}$ and $\psi \in \text{Aut}(N_k((O_X)^{+0}))$.

Each permutation on the set $\mathbb{A} \setminus \{z\}$ and each permutation on the set $\mathbb{B} \setminus \{0, |z_0^z\}$ define the automorphism $\psi : N_k((O_X)^{+0}) \rightarrow N_k((O_X)^{+0})$. Indeed, $\psi(0 * \mathcal{L}) = \psi(0) = 0 = 0 * \psi(\mathcal{L}) = \psi(0) * \psi(\mathcal{L})$, $\psi(\mathcal{L} * 0) = \psi(0) = 0 = \psi(\mathcal{L}) * 0 = \psi(\mathcal{L}) * \psi(0)$, $\psi(\mathcal{A}_1 * \mathcal{A}_2) = \psi(z) = z = \psi(\mathcal{A}_1) * \psi(\mathcal{A}_2)$ for any $\mathcal{L} \in N_k((O_X)^{+0})$, $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{A}$, and $\psi(\mathcal{B} * \mathcal{L}) = \psi(|z_0^z) = |z_0^z = \psi(\mathcal{B}) * \psi(\mathcal{L})$, $\psi(\mathcal{L} * \mathcal{B}) = \psi(|z_0^z) = |z_0^z = \psi(\mathcal{L}) * \psi(\mathcal{B})$ for any $\mathcal{B} \in \mathbb{B} \setminus \{0\}$, $\mathcal{L} \in N_k((O_X)^{+0}) \setminus \{0\}$.

Therefore, $\text{Aut}(N_k((O_X)^{+0})) \cong S_{\mathbb{A} \setminus \{z\}} \times S_{\mathbb{B} \setminus \{0, |z_0^z\}}$. \square

The set \mathbb{A} from Theorem 4.7 can be identified with the subsemigroup $N_k(O_X)$ of the semigroup $N_k((O_X)^{+0})$. Consequently, for finite null semigroups O_n , we have the following corollary.

Corollary 4.8. *For each $k \geq 2$ and $n \geq 2$, the automorphism group of the semigroup $N_k((O_n)^{+0})$ is isomorphic to the group $S_{|N_k(O_n)|-1} \times S_{|N_k((O_n)^{+0})|-|N_k(O_n)|-3}$.*

Using the results of Table 1 and Corollary 4.8 in Table 8, we present the automorphism groups of the semigroups $N_k((O_n)^{+0})$ for $k \in \{2, 3, 4\}$ and $n \leq 4$.

n	$\text{Aut}(N_2((O_n)^{+0}))$	$\text{Aut}(N_3((O_n)^{+0}))$	$\text{Aut}(N_4((O_n)^{+0}))$
1	C_1	C_1	C_1
2	$C_2 \times S_5$	$C_2 \times S_4$	$C_2 \times S_4$
3	$S_{10} \times S_{66}$	$S_9 \times S_{41}$	$S_9 \times S_{40}$
4	$S_{79} \times S_{2562}$	$S_{53} \times S_{705}$	$S_{52} \times S_{631}$

Table 8: The automorphism groups of the semigroups $N_k((O_n)^{+0})$ for $k \in \{2, 3, 4\}$ and $n \leq 4$

5. Acknowledgment

The author would like to express his sincere thanks to the anonymous referees for a very careful reading of the paper and for all their insightful comments and valuable suggestions, which improve considerably the presentation of this paper.

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